Uniqueness of equilibrium with survival probability heterogeneity and endogenous annuity price

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Abstract

When annuitants' survival probabilities are heterogeneous, the equilibrium annuity price is affected by their annuitization choices, which further depend on the annuity price. Given this mutual dependence, it is generally difficult to establish uniqueness of the equilibrium. Based on similar expressions appearing in several annuity and insurance models, we obtain two results in an annuity model with heterogeneity in survival probability only. First, the equilibrium annuity price is always unique if the annuitization function is multiplicatively separable in survival probability and annuity price. Second, the equilibrium is unique for more general annuitization functions, provided that a sufficient condition on the distribution of survival probabilities holds. Many distributions, including the uniform and normal distributions, satisfy this condition.

Keywords: survival probability heterogeneity; annuitization choices; annuity price; mutual dependence; uniqueness of equilibrium

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1 Introduction

The concept of equilibrium is widely used in economic analysis, as it allows us to systematically explain and predict complex human behavior. Moreover, sharper predictions usually arise when the equilibrium is unique. The uniqueness of equilibrium is important in many theoretical and applied studies.

This paper considers the equilibrium annuity price when it is endogenously determined in an environment of heterogeneous survival probabilities. While the existence of the equilibrium annuity price has been shown in many papers in the literature, establishing its uniqueness turns out to be difficult. Using an overlapping-generations model with annuities and public pension, Abel (1986, p. 1086) mentions that multiple equilibria cannot be ruled out unless the utility function is logarithmic. In a three-period life-cycle model analyzing deferred and immediate annuities, Brugiavini (1993, p. 43) points out that in general there exists at least one root to the equation defining the equilibrium of the immediate annuity market. Regarding the issue of uniqueness, she only mentions that there is a unique value for the equilibrium rate of return when preferences are specified in the logarithmic form (p. 60). In a model analyzing both annuity and life insurance, Villeneuve (2003, p. 534) shows that there exists an equilibrium in the annuity market, but "uniqueness is not warranted."

The difficulty of establishing uniqueness of equilibrium in the above models with heterogeneous survival probabilities comes from the mutual dependence of annuitization choices and the annuity price: the equilibrium annuity price is determined by the buyers' annuitization choices, which further depend on the annuity price (and their survival probabilities). In understanding the source of difficulty of the uniqueness proof, we observe that there is a common expression that determines the equilibrium annuity price in various models, such as Abel (1986), Brugiavini (1993), Villeneuve (2003) and Lau and Zhang (2023). This motivates us to focus on a particular function, given by (10) in Section 2, when we study uniqueness issues in annuity models.

To illustrate the underlying idea of our proof as clearly as possible, we keep only the essential features from various papers and consider a simple model of the annuity market, in which the equilibrium annuity price is determined by the fixed point of a function that is similar to the corresponding equations in the above-mentioned papers. We obtain two main results in this paper. First, we show in Proposition 1 that the equilibrium annuity price is always unique if the annuitization function, defined in (4) in Section 2, is multiplicatively separable in survival probability and annuity price. This result covers two well-known cases in the literature: the immediate annuity model with logarithmic utility function (Abel, 1986; Brown, 2003) and the deferred annuity model in which the annuities are purchased in an early period before the buyers' survival probability information is revealed (Brugiavini, 1993). Second, we consider more general annuitization functions that may not be multiplicatively separable. In this case, the proof is much more complicated, and we transform the uniqueness condition to a more convenient form, as (21) in Section 3. We then show in Proposition 2 that the equilibrium annuity price is unique for all time-separable utility functions, subject to a sufficient condition on the probability density function of the annuitants' survival probabilities. Various commonly-used distributions such as the uniform distribution and the normal distribution (with appropriate truncation and parameter restrictions) satisfy this condition. Our results suggest that uniqueness of the equilibrium can be established in annuity models with a wide class of probability distributions.

It is puzzling why the sufficient condition for the uniqueness of equilibrium annuity price in Proposition 2 is expressed in terms of the survival probability distribution only, but not other economic factors. We investigate this question by linking the slope of the function determining the equilibrium and the derivative of the annuity providers' budget balance with respect to the annuity price. It is shown that a change in annuity price has a positive direct effect on the budget balance for all annuitants but a negative indirect effect (through changing annuitization choices) for those with low survival probabilities. According to this interpretation, the sufficient condition for the uniqueness of equilibrium puts an upper bound on the indirect effect for the low-risk group.

The rest of this paper is organized as follows. In Section 2, we introduce a simple annuity model and characterize the equilibrium annuity price under the zero-profit condition. In Section 3, we show that the equilibrium annuity price is always unique when the annuitization function is multiplicatively separable in survival probability and annuity price. In Section 4, we examine the more general case that the annuitization function is not multiplicatively separable, and obtain a sufficient condition for uniqueness of equilibrium. We provide concluding remarks in Section 5.

2 A simple model of the annuity market

There are a variety of annuity models with asymmetric information (as in Brugiavini, 1993; Brown, 2003; Villeneuve, 2003; Steinorth, 2012; Lau and Zhang, 2023), incorporating different features such as a single contract versus a menu of contracts, exclusive versus non-exclusive contracts, and pure life annuity versus annuities with guarantees or cost-of-living adjustment, etc. Moreover, researchers represent survival probability heterogeneity as either a continuous or discrete random variable. It is impossible to incorporate all relevant features in one model. Observing a similar expression of the equilibrium condition in several papers in the literature, we choose a related model of annuitization with survival probability heterogeneity to study uniqueness issues of the equilibrium annuity price.

Three features of this model are highlighted. First, there is a continuum of annuitants indexed by their private information of the probability of surviving to the second period (θ). This private information is the source of adverse selection, in which the annuitants with longer expected lifetime demand more annuities. Second, instead of exclusive contracts with price convexity (as in Eichenbaum and Peled, 1987; Steinorth, 2012), it is assumed that the annuity providers offer the annuitants a non-exclusive financial contract with linear pricing (as in Abel, 1986; Brugiavini, 1993; Hosseini, 2015). The assumption of non-exclusive annuity contracts with linear pricing, in which a typical annuity provider specifies the unit price of an annuity and allows the annuitants to choose the amount of purchase, is more in line with observed practices.[1] Third, the equilibrium price of the annuity is determined by a zero-profit condition, as in Abel (1986), Villeneuve (2003) and Hosseini (2015). This condition is usually justified on the basis of the assumption of free entry and exit of annuity companies.

The key elements of the two-period model of annuitization behavior in the presence of survival probability heterogeneity capture the idea present in various models in the literature, such as Abel (1986), Brugiavini (1993), Brown (2003), Villeneuve (2003), Lockwood (2012), and Lau and Zhang (2023). Our proposed uniqueness proof is most clearly seen in this simple environment, in which we do not consider other elements such as the public sector (Abel, 1986), deferred annuities (Brugiavini, 1993), life insurance (Villeneuve, 2003) and bequest motive (Lockwood, 2012; Lau and Zhang, 2023).

[1] More generally, non-exclusive trade is observed in various markets such as the security or insurance market (Attar et al., 2011). On the other hand, the effectiveness of exclusive contracts in the annuity market has been questioned in the literature. Abel (1986) argues that the specification of exclusive contracts may not be an appropriate assumption for the annuity market because it is difficult to determine whether a buyer also holds annuities from other providers. A similar point has also been mentioned by Cawley and Philipson (1999, p. 831): "several small contracts would be cheaper than a large one under convex pricing."

2.1 Annuitization choices

In the model, there are a continuum of annuitants who have just retired and live for two periods at most: Period 1 with certainty and Period 2 with some probability. The two periods correspond to, respectively, the early and advanced stages of retirement. The annuitants have different probabilities of surviving to Period 2, represented by a probability density function $f(\theta)$, where $\theta \in [\theta^L, \theta^H]$ represents survival probability and $0 \leq \theta^L < \theta^H \leq 1$.

The annuity contract operates as follows. It is offered in Period 1, and the price of one unit of annuity is p. If an annuitant buys one unit of annuity in Period 1, she will receive one dollar in Period 2 if and only if she is alive.[2] With this annuity product, the budget constraints of the annuitants who survive to Period 2 are given by

$$c_1 = w - p\alpha, \tag{1}$$

and

$$c_2 = \alpha, \tag{2}$$

where c_i is the level of consumption expenditure in Period i $(i = 1, 2)$, α is the amount of annuity purchase and w is the retirement wealth, which is assumed to be the same for every annuitant in this model.

It is assumed that an annuitant with survival probability θ has time-separable utility function

$$U(c_1, c_2; \theta) = u(c_1) + \theta\delta u(c_2), \tag{3}$$

where δ $(\delta \leq 1)$ is the subjective discount factor. Moreover, the standard assumptions that $u(c)$ is strictly concave and $\lim_{c\to 0} u^{'}(c) = \infty$ hold.

Combining (1), (2) and (3), the annuitant's decision problem becomes

$$\max_{\alpha} \left[u(w - p\alpha) + \theta\delta u(\alpha) \right].$$

It is straightforward to show that the optimal annuity choice of a buyer with survival probability θ when facing annuity price p, denoted by $\alpha(\theta, p)$, is an interior solution in the interval $(0, w)$ and is defined by

$$pu'(w - p\alpha(\theta, p)) = \theta\delta u'(\alpha(\theta, p)). \tag{4}$$

[2] We define the annuity contract in terms of the annuity price (p), as in Brown (2003). In this case, the annuity payout is fixed at the normalized level of 1. Alternatively, it can be defined in terms of the annuity payout level, as in Abel (1986), Brugiavini (1993) and Lockwood (2012). The two specifications are equivalent, with the annuity price negatively related to the annuity payout.

This optimal choice is unique.[3] Moreover,

$$\frac{\partial \alpha(\theta, p)}{\partial \theta} = \frac{-\delta u'(\alpha(\theta, p))}{p^2 u''(w - p\alpha(\theta, p)) + \theta \delta u''(\alpha(\theta, p))} > 0. \tag{5}$$

The annuity purchase and survival probability of the annuitants are positively related. The positive correlation is similar to (11) of Abel (1986).

2.2 The equilibrium relationship under the zero-profit assumption

The next step is to determine the equilibrium value of the annuity price in this model. When the annuity price is p, the total amount of annuity premium received by the annuity providers at Period 1 is $\int_{\theta^L}^{\theta^H} p\alpha(\theta, p) f(\theta) d\theta$. Together with investment return, the corresponding value in Period 2 given by

$$R \int_{\theta^L}^{\theta^H} p\alpha(\theta, p) f(\theta) d\theta, \tag{6}$$

where $\alpha(\theta, p)$ is defined in (4) and R ($R \geq 1$) is the gross interest rate of risk-free bond.

Since the annuitants have different probabilities θ to survive in Period 2 and receive the annuity payment, the annuity providers' total expected payment to the surviving annuitants is

$$\int_{\theta^L}^{\theta^H} \theta\alpha(\theta, p) f(\theta) d\theta. \tag{7}$$

We assume the zero-profit condition for annuity provision. Under this assumption, the equilibrium price, denoted by p^*, is obtained by equating (6) and (7).[4] This leads to the relationship

$$p^* = \frac{\int_{\theta^L}^{\theta^H} \theta\alpha(\theta, p^*) f(\theta) d\theta}{R \int_{\theta^L}^{\theta^H} \alpha(\theta, p^*) f(\theta) d\theta}. \tag{8}$$

[3] The left-hand side in (4) is strictly increasing in $\alpha(\theta, p)$ while the right-hand side is strictly decreasing in $\alpha(\theta, p)$. These features guarantee a unique value of optimal $\alpha(\theta, p)$ at a given level of annuity price p.

[4] Similar to Abel (1986), Brugiavini (1993), Villeneuve (2003) and Lockwood (2012), we focus on the risk-sharing feature of the annuity market and do not emphasize administrative costs in annuity provision. Thus, administrative cost does not appear in the zero-profit condition equating (6) and (7), leading to the well-known result that the return of holding the risk-free bond is dominated by that of holding the annuity (Yaari, 1965; Davidoff et al., 2005). As a result, the risk-free bond, which is relevant for the annuity providers, is absent in the annuitant's maximization problem when it is assumed that there is no bequest motive in the annuitant's objective function (3).

According to the above analysis, the buyers' annuitization choices depend on the annuity price, as in (4); at the same time, the annuity price depends on the annuitization choices of different buyers under the zero-profit condition. The mutual dependence of $\alpha(\theta, p)$ and p is clearly seen when we express the equilibrium annuity price (p^*) in (8) as the fixed point of the following function:

$$J(p^*) = p^*, \tag{9}$$

where $J(p)$ is defined by

$$J(p) = \frac{\int_{\theta^L}^{\theta^H} \theta\alpha(\theta, p) f(\theta) d\theta}{R\int_{\theta^L}^{\theta^H} \alpha(\theta, p) f(\theta) d\theta} = \frac{\frac{1}{R}\int_{\theta^L}^{\theta^H} \theta\alpha(\theta, p) f(\theta) d\theta}{\int_{\theta^L}^{\theta^H} \alpha(\theta, p) f(\theta) d\theta}. \tag{10}$$

The $J(p)$ function in (10) is constructed on the basis of (6) and (7). The denominator on the right-hand side of (10) is the total amount of annuities purchased in Period 1, and the numerator is the present discounted value (measured in Period 1) of the expected amount of payment by the annuity providers to annuitants who are alive in Period 2. Both terms are calculated at an arbitrary value of p.

The $J(p)$ function is useful in subsequent analysis, because the relative size of p and the ratio in $J(p)$ determines whether the overall budget of the annuity providers is in balance or not. It can easily be seen that when this ratio is smaller than p, the total value of annuity revenue is higher than the total value of payment, leading to a surplus for the annuity providers as a whole. On the other hand, the overall budget of the annuity providers is in deficit when this ratio is larger than p. The overall budget of all annuity providers is in balance when (9) holds.

2.3 Uniqueness of the equilibrium annuity price

We now examine the function $J(p)$ in (10) and look for the intersection of this function and the 45-degree line when $p \in \left[\frac{E(\theta)}{R}, \frac{\theta^H}{R}\right]$.[5] First, it can be shown that

$$J\left(\frac{E(\theta)}{R}\right) > \frac{E(\theta)}{R} \tag{11}$$

at the beginning point $p = \frac{E(\theta)}{R}$, because of the positive correlation between annuitization choice and survival probability according to (5).[6] When adverse

[5] The function $J(p)$ is well-defined for all values of p in this range, because $\alpha(\theta, p) > 0$ for all $\theta > 0$ according to (4).

[6] Formally, we have $RJ(p) = \frac{\int_{\theta^L}^{\theta^H} \theta\alpha(\theta,p)f(\theta)d\theta}{\int_{\theta^L}^{\theta^H} \alpha(\theta,p)f(\theta)d\theta} = \frac{E(\theta\alpha(\theta,p))}{E(\alpha(\theta,p))} = \frac{cov(\theta,\alpha(\theta,p))}{E(\alpha(\theta,p))} + E(\theta) > E(\theta)$ for any value of p, because of (5). Substituting $p = \frac{E(\theta)}{R}$ leads to (11).

selection is present, the overall budget of annuity providers is in deficit if the annuity price is set at the actuarially fair level of $\frac{E(\theta)}{R}$.

Second, consider the end point $p = \frac{\theta^H}{R}$. It is easy to show that

$$J\left(\frac{\theta^H}{R}\right) = \frac{\int_{\theta^L}^{\theta^H} \theta\alpha\left(\theta, \frac{\theta^H}{R}\right) f(\theta)d\theta}{R\int_{\theta^L}^{\theta^H} \alpha\left(\theta, \frac{\theta^H}{R}\right) f(\theta)d\theta} < \frac{\int_{\theta^L}^{\theta^H} \theta^H\alpha\left(\theta, \frac{\theta^H}{R}\right) f(\theta)d\theta}{R\int_{\theta^L}^{\theta^H} \alpha\left(\theta, \frac{\theta^H}{R}\right) f(\theta)d\theta} = \frac{\theta^H}{R}, \quad (12)$$

because $\theta < \theta^H$ for $\theta \in \left[\theta^L, \theta^H\right)$. The overall budget of annuity providers is in surplus if the price is at the end point $p = \frac{\theta^H}{R}$.

According to (11) and (12), $J(p)$ is above the 45-degree line at the starting point $\frac{E(\theta)}{R}$ but below the line at the end point $\frac{\theta^H}{R}$. Since the $J(p)$ function is continuous, we conclude that the equilibrium annuity price p^* exists within the interval $\left(\frac{E(\theta)}{R}, \frac{\theta^H}{R}\right)$, as shown in Figure 1. Moreover, p^* is unique if

$$J'(p^*) < 1. \quad (13)$$

If the slope of the $J(p)$ function at the equilibrium p^* is less than 1, it is not possible for the $J(p)$ function to intersect the 45-degree line from below, after intersecting it from above for the first time (which is guaranteed, as seen in the existence proof above).[7]

[Insert Figure 1 here.]

In the following analysis, we use (13) to obtain two main results about the uniqueness of equilibrium annuity price. In Section 3, we first consider the simpler case in which the annuitization function $\alpha(\theta, p)$, defined in (4), is multiplicatively separable in θ and p. In Section 4, we consider the equilibrium uniqueness issues for more general annuitization functions.

3 Multiplicatively separable annuitization function

It is well known in the literature that there are two special cases in which the equilibrium annuity price is unique. We now use the framework in the previous section to generalize these results.

In the first special case, the within-period utility function is logarithmic:

$$u(c) = \ln c, \quad (14)$$

[7]See also Stokey et al. (1989, pp. 50-51) for similar points.

as in Appendix B of Abel (1986). He uses an overlapping-generations model with uncertain lifetimes to examine the effects of adverse selection on the pricing of private annuities and on annuitants' behavior, and shows that the equilibrium of the annuity market is unique in the case of logarithmic utility function, but multiple equilibria cannot be ruled out for more general utility functions. The same result also appears in the model in Section 2. With the logarithmic utility function (14), it is straightforward to show that (4) becomes

$$\alpha(\theta, p) = \frac{\delta w \theta}{(1 + \delta\theta) p}. \tag{15}$$

The annuitization function in (15) is an example of the class of multiplicatively separable functions:[8]

$$\alpha(\theta, p) = \alpha_1(\theta)\, \alpha_2(p). \tag{16}$$

When $\alpha(\theta, p)$ is given by (16), the corresponding $J(p)$ function is

$$J(p) = \frac{\int_{\theta^L}^{\theta^H} \theta\alpha_1(\theta)\,\alpha_2(p)\, f(\theta)d\theta}{R\int_{\theta^L}^{\theta^H} \alpha_1(\theta)\,\alpha_2(p)\, f(\theta)d\theta} = \frac{\int_{\theta^L}^{\theta^H} \theta\alpha_1(\theta)\, f(\theta)d\theta}{R\int_{\theta^L}^{\theta^H} \alpha_1(\theta)\, f(\theta)d\theta}, \tag{17}$$

which is independent of p.

The second special case appears in Brugiavini's (1993) influential study, in which she uses a three-period model with longevity risk only to understand the role of uncertainty resolution on annuitization behavior. In that model, the annuitants' health characteristics are identical at an early age but different at an advanced age. When the equilibrium price of deferred annuities is determined by the zero-profit condition, the corresponding annuitization function is a special example of (16) with $\alpha_1(\theta) = 1$, because the amount of deferred annuities purchased by all individuals at the early period are identical.[9] When $\alpha_1(\theta) = 1$, the corresponding $J(p)$ function can be simplified to

$$J(p) = \frac{\int_{\theta^L}^{\theta^H} \theta\alpha_2(p)\, f(\theta)d\theta}{R\int_{\theta^L}^{\theta^H} \alpha_2(p)\, f(\theta)d\theta} = \frac{\int_{\theta^L}^{\theta^H} \theta f(\theta)d\theta}{R\int_{\theta^L}^{\theta^H} f(\theta)d\theta} = \frac{E(\theta)}{R}. \tag{18}$$

The buyer's annuitization function in the immediate annuity model with logarithmic utility function (as in Abel, 1986; Brown, 2003) and that in the deferred annuity model with identical early health characteristics (as

[8] The specification of multiplicatively separable functions has been used in many studies in economics and finance, such as Coeurdacier et al. (2015) and Babenko et al. (2016).

[9] This special case of $\alpha_1(\theta) = 1$ also appears if the annuitization function $\alpha(\theta, p)$ is mandated to be identical for all buyers by the government.

in Brugiavini, 1993) are two special cases of the multiplicatively separable function (16). In each of these two cases, the corresponding $J(p)$ function is independent of p, and

$$J'(p) = 0 \tag{19}$$

for all value of p. Therefore, we conclude from (13) that the equilibrium annuity price is unique.

We state the above results in the following proposition.

Proposition 1. *Consider a two-period model in which the annuitants' survival probabilities are heterogeneous and their decision problems are given by (1) to (3). If the annuitization function is multiplicatively separable in survival probability and annuity price, as in (16), then the equilibrium annuity price (p^*) is unique.*

Proposition 1 shows that if the underlying economic factors, such as the form of utility function (in the first special case) or information structure (identical early health signal in the second special case), lead to a multiplicatively separable annuitization function, then the equilibrium annuity price is always unique. The intuition of this proposition can be traced to the ratio of the present discounted value of expected payment (measured in Period 1) to the total amount of purchased annuities, given by the $J(p)$ function in (10). When the annuitization function $\alpha(\theta, p)$ defined in (4) is multiplicatively separable in θ and p, it is straightforward to see from (17) that both the numerator and denominator of the $J(p)$ function are proportional to $\alpha_2(p)$. Thus, the effects of the annuity price (p) on the numerator and denominator terms exactly cancel out, leading to a horizontal $J(p)$ function. The fixed point of the horizontal $J(p)$ function is always unique.

4 More general annuitization functions

Proposition 1 is applicable when the annuitization function $\alpha(\theta, p)$ is multiplicatively separable in θ and p. However, a buyer's annuitization choice generally depends on θ and p in a complicated manner,[10] and (19) may not hold. We now consider uniqueness issues in this general case.

When the annuitization function is more general and not multiplicatively separable, the following proposition states a sufficient condition for the uniqueness of equilibrium annuity price. The proof of this proposition is

[10] For example, when the within-period utility function is CRRA: $u(c) = \frac{c^{1-\phi}-1}{1-\phi}$, it is straightforward to show that the annuitization function $\alpha(\theta, p)$ is not multiplicatively separable when $\phi \neq 1$.

more complicated than that of Proposition 1. After presenting the proof, we will summarize the key underlying idea behind the proof.

Proposition 2. *Consider a two-period model in which the annuitants' survival probabilities are heterogeneous and their decision problems are given by (1) to (3). If the probability density function of the annuitants' survival probabilities satisfies the condition that*

$$\frac{d\left[\theta f\left(\theta\right)\right]}{d\theta}=f(\theta)+\theta f'(\theta)\geq 0 \tag{20}$$

for all $\theta\in\left[\theta^{L},\theta^{H}\right]$, then the equilibrium annuity price (p^{*}) is unique.

Proof:

Differentiating (10) with respect to p, we obtain

$$J'\left(p\right)=\frac{\int_{\theta^{L}}^{\theta^{H}}\theta\frac{\partial\alpha(\theta,p)}{\partial p}f(\theta)d\theta}{R\int_{\theta^{L}}^{\theta^{H}}\alpha\left(\theta,p\right)f(\theta)d\theta}-\frac{J\left(p\right)\int_{\theta^{L}}^{\theta^{H}}\frac{\partial\alpha(\theta,p)}{\partial p}f(\theta)d\theta}{\int_{\theta^{L}}^{\theta^{H}}\alpha\left(\theta,p\right)f(\theta)d\theta},$$

which holds for all values of p. At the equilibrium value p^{*} defined by (9), it can be shown that (13) is equivalent to

$$\frac{\int_{\theta^{L}}^{\theta^{H}}\theta\frac{\partial\alpha(\theta,p^{*})}{\partial p^{*}}f(\theta)d\theta-Rp^{*}\int_{\theta^{L}}^{\theta^{H}}\frac{\partial\alpha(\theta,p^{*})}{\partial p^{*}}f(\theta)d\theta}{R\int_{\theta^{L}}^{\theta^{H}}\alpha\left(\theta,p^{*}\right)f(\theta)d\theta}<1,$$

which can further be shown to be equivalent to

$$\int_{\theta^{L}}^{\theta^{H}}K\left(\theta,p^{*}\right)f(\theta)d\theta>0, \tag{21}$$

where

$$K\left(\theta,p\right)=R\alpha\left(\theta,p\right)+\left(Rp-\theta\right)\frac{\partial\alpha\left(\theta,p\right)}{\partial p}. \tag{22}$$

Differentiating (4) with respect to p, we obtain

$$\frac{\partial\alpha\left(\theta,p\right)}{\partial p}=\frac{u'\left(w-p\alpha\left(\theta,p\right)\right)-p\alpha\left(\theta,p\right)u''\left(w-p\alpha\left(\theta,p\right)\right)}{p^{2}u''\left(w-p\alpha\left(\theta,p\right)\right)+\theta\delta u''\left(\alpha\left(\theta,p\right)\right)}<0. \tag{23}$$

In analyzing whether (21) holds or not, we separate $\left[\theta^{L},\theta^{H}\right]$ into 2 intervals: $\left[\theta^{L},Rp^{*}\right]$ and $\left(Rp^{*},\theta^{H}\right]$. As shown in subsequent analysis, the annuitants in these two intervals can be interpreted as belonging to the low-risk and high-risk groups, respectively.

For the high-risk group with $\theta \in \left(Rp^*, \theta^H\right]$, we have $\theta > Rp^*$. Combining with (23), we conclude that $K(\theta, p^*) > 0$ according to (22). As a result,

$$\int_{Rp^*}^{\theta^H} K(\theta, p^*) f(\theta) d\theta > 0. \tag{24}$$

For the low-risk group with $\theta \in \left[\theta^L, Rp^*\right]$, we have $\theta < Rp^*$. Thus, $R\alpha(\theta, p^*)$ is positive and $(Rp^* - \theta) \frac{\partial \alpha(\theta, p^*)}{\partial p}$ is negative, leading to an ambiguous sign of $K(\theta, p^*)$ generally according to (22). Using (5), (22) and (23), we obtain

$$\begin{aligned} K(\theta, p) = (\theta - Rp)\left(\frac{\theta}{p}\right)\frac{\partial \alpha(\theta, p)}{\partial \theta} \\ +\theta\alpha(\theta, p)\frac{pu''(w - p\alpha(\theta, p)) + R\delta u''(\alpha(\theta, p))}{p^2 u''(w - p\alpha(\theta, p)) + \theta\delta u''(\alpha(\theta, p))}, \end{aligned} \tag{25}$$

which holds for all values of p.[11] When $\theta \in \left[\theta^L, Rp^*\right]$, we have

$$\frac{(p^*)^2 u''(w - p^*\alpha(\theta, p^*)) + Rp^*\delta u''(\alpha(\theta, p^*))}{(p^*)^2 u''(w - p^*\alpha(\theta, p^*)) + \theta\delta u''(\alpha(\theta, p^*))} > 1. \tag{26}$$

Combining (25) and (26), we obtain

$$\begin{aligned} &\int_{\theta^L}^{Rp^*} K(\theta, p^*) f(\theta) d\theta \\ &> \int_{\theta^L}^{Rp^*} \left[(\theta - Rp^*)\left(\frac{\theta}{p^*}\right)\frac{\partial \alpha(\theta, p^*)}{\partial \theta} + \left(\frac{\theta}{p^*}\right)\alpha(\theta, p^*)\right] f(\theta) d\theta \\ &= \frac{1}{p^*}\int_{\theta^L}^{Rp^*} \theta^2 \frac{\partial \alpha(\theta, p^*)}{\partial \theta} f(\theta) d\theta - R\int_{\theta^L}^{Rp^*} \theta \frac{\partial \alpha(\theta, p^*)}{\partial \theta} f(\theta) d\theta + \frac{1}{p^*}\int_{\theta^L}^{Rp^*} \theta\alpha(\theta, p^*) f(\theta) d\theta \\ &= \frac{1}{p^*}\left\{\left(Rp^* - \theta^L\right)\alpha\left(\theta^L, p^*\right)\theta^L f(\theta^L) + \int_{\theta^L}^{Rp^*} (Rp^* - \theta)\alpha(\theta, p^*)\left[f(\theta) + \theta f'(\theta)\right] d\theta\right\}, \end{aligned} \tag{27}$$

after applying integration by parts to the first and second terms in the third line of (27). The first term in the last line of (27) is zero or positive. If (20) holds, then the second term in the last line of (27) is zero or positive.[12]

[11] Standard (but tedious) procedure to derive (25) and some other equations can be found in the Online Appendix.

[12] As seen in (27), we only need condition (20) to be satisfied in the interval $\left[\theta^L, Rp^*\right]$. However, p^* is determined endogenously, depending on $f(\theta)$ and other factors such as the annuitant's utility function and the market interest rate. Proposition 2 specifies sufficient condition (20) for the whole interval $\left[\theta^L, \theta^H\right]$, so that one could check whether it holds or not irrespective of other components of the model.

Together with (24), we conclude that (21), or equivalently (13), holds when (20) is satisfied. This proves Proposition 2. ■

To summarize, there are three main steps in the above proof. First, condition (13) for uniqueness is equivalent to (21). Second, $\int_{Rp^*}^{\theta^H} K(\theta, p^*) f(\theta) d\theta > 0$. Third, a sufficient condition for $\int_{\theta^L}^{Rp^*} K(\theta, p^*) f(\theta) d\theta \geq 0$ to hold is (20), as shown in (27).

4.1 The idea behind the proof of Proposition 2

The following decomposition is helpful to understand the intuition of condition (20) and the reasons behind the proof of Proposition 2. Combining (6) and (7), we define

$$\pi(p) = R \int_{\theta^L}^{\theta^H} p\alpha(\theta, p) f(\theta) d\theta - \int_{\theta^L}^{\theta^H} \theta\alpha(\theta, p) f(\theta) d\theta, \tag{28}$$

which expresses the annuity providers' budget balance as a function of annuity price. Differentiating (28) with respect to p, we have

$$\pi'(p) = R \int_{\theta^L}^{\theta^H} \alpha(\theta, p) f(\theta) d\theta + \int_{\theta^L}^{\theta^H} (Rp - \theta) \frac{\partial \alpha(\theta, p)}{\partial p} f(\theta) d\theta. \tag{29}$$

As observed in (28), both the revenue and expected payment of the annuity providers depend on annuitants' choices $\alpha(\theta, p)$, and the revenue also depends directly on the price (p) paid for each unit of annuity purchase. Accordingly, there is a useful way to decompose the effect of a change in annuity price on the annuity providers' budget balance: the *direct effect* on the revenue, and the *indirect effect* on the revenue and payment through changing annuitization behavior. The first term on the right-hand side of (29), which does not take into account the induced changes in $\alpha(\theta, p)$ and is always positive, represents the direct effect. On the other hand, the indirect effect on the budget balance through the change in $\alpha(\theta, p)$ is given by the second term on the right-hand side of (29). Comparing (29) with (21) and (22), we conclude that (21) is equivalent to

$$\pi'(p^*) > 0, \tag{30}$$

which has an interpretation that the derivative of the budget balance with respect to annuity price at the equilibrium (when $p = p^*$) is positive.

Pursuing the above line of thought, (22) represents the decomposition into direct and indirect effects at the individual level (for an annuitant with survival probability θ). It is helpful to examine the low-risk and high-risk groups separately. We show in the proof of (24) that the indirect effect is positive for the *high-risk group*, defined as those with $\theta \in \left(Rp^*, \theta^H\right]$. The underlying reason is as follows. An increase in p leads to lower annuitization according to (23), and thus to a decrease in both revenue and payment for the annuity providers. Since high-risk annuitants ($\theta > Rp^*$) are more likely to survive to Period 2, the magnitude of the expected decrease in payment ($-\theta \frac{\partial \alpha(\theta,p^*)}{\partial p}$) is larger than the magnitude of the decrease in revenue ($-Rp^* \frac{\partial \alpha(\theta,p^*)}{\partial p}$). As a result, the indirect effect $(Rp^* - \theta) \frac{\partial \alpha(\theta,p^*)}{\partial p} > 0$ for each high-risk annuitant.

The analysis for the *low-risk group*, when $\theta \in \left[\theta^L, Rp^*\right]$, is more challenging. For this group, the magnitude of the expected decrease in payment is smaller than that of the decrease in revenue. Thus, the indirect effect is negative. Through the link of $\frac{\partial \alpha(\theta,p)}{\partial p}$ and $\frac{\partial \alpha(\theta,p)}{\partial \theta}$, and the use of integration by parts, we show in (27) that the sum of direct and indirect effects is nonnegative if the sufficient condition (20) is satisfied.[13] Condition (20), which is about the probability density function of the survival probability, ensures that the negative indirect effect is bounded for the low-risk group. To understand the intuition of this condition, first note that the area under the probability density function $f(\theta)$ is 1 by definition. Condition (20) always holds when $f(\theta)$ is upward sloping or horizontal. In these cases, the proportion of annuitants with low risk is not high, leading to a non-negative total effect. The problematic case may occur when $f(\theta)$ is downward-sloping and the weight of annuitants with low survival probabilities is high. In particular, the annuitants with survival probabilities close to θ^L are likely to have a strong influence on the magnitude of the negative indirect effect, given by $(Rp^* - \theta) \frac{\partial \alpha(\theta,p^*)}{\partial p}$ in (22), because the value of the first component is highest at $\theta = \theta^L$. The restriction $\frac{d[\theta f(\theta)]}{d\theta} = f(\theta) + \theta f'(\theta) \geq 0$ in (20) puts an upper bound on the weight of the low-risk group by preventing $f(\theta)$ to be too steeply downward-sloping. When condition (20) is satisfied, the weight of the low-risk group is not high enough to cause the sum of direct and indirect effects to be negative for the low-risk group, according to (27). As a result, (21) holds and the equilibrium annuity price is unique.

[13] The connection between $\frac{\partial \alpha(\theta,p)}{\partial p}$ (how a change in annuity price affects annuitization choice) in (23) and $\frac{\partial \alpha(\theta,p)}{\partial \theta}$ (how a change in survival probability affects annuitization choice) in (5) is traced to the procedure that each of the two terms is obtained by differentiating the same first-order condition (4). This link is important because it allows (22) to be expressed in an equivalent form (25) that can easily be integrated with respect to θ, as required by (21).

4.2 Probability density functions satisfying Proposition 2: Some applications

We show in Proposition 2 that the equilibrium annuity price is unique if condition (20) on the survival probability distribution holds. However, the result may not be very useful if this condition is restrictive. We now examine whether this condition is satisfied or not for several probability density functions frequently used in the study of annuitization behavior.

In the two-period model, we assume that $\theta \in \left[\theta^L, \theta^H\right]$, where $0 \leq \theta^L < \theta^H \leq 1$. We first focus on the uniform distribution, with the probability density function

$$f(\theta) = \frac{1}{\theta^H - \theta^L}. \tag{31}$$

Since $f'(\theta) = 0$, it is straightforward to conclude that (20) holds for the uniform distribution.

Another commonly-used distribution, particularly when the random variable exhibits the bell-shaped feature, is the normal distribution, where $\theta \in (-\infty, \infty)$. Since the normal distribution is not within a bounded interval, we need to include an additional step of truncating them within the interval $\left[\theta^L, \theta^H\right]$. To fit $\theta \in \left[\theta^L, \theta^H\right]$, the probability density function of the truncated normal distribution is given by[14]

$$f(\theta) = \frac{e^{-\frac{1}{2}\left(\frac{\theta-\mu}{\sigma}\right)^2}}{\int_{\theta^L}^{\theta^H} e^{-\frac{1}{2}\left(\frac{\theta-\mu}{\sigma}\right)^2} d\theta}, \tag{32}$$

where μ and σ^2 are the mean and variance of θ before truncation. It can be shown from (32) that

$$f(\theta) + \theta f'(\theta) = \left[1 - \frac{\theta\left(\theta - \mu\right)}{\sigma^2}\right] f(\theta).$$

Since $1 - \frac{\theta(\theta-\mu)}{\sigma^2}$ is decreasing in θ, its smallest value occurs at $\theta = \theta^H$. We conclude that if

$$\sigma^2 \geq \theta^H \left(\theta^H - \mu\right), \tag{33}$$

then (20) holds for the truncated normal distribution (32). Condition (33) means that the normal distribution has to be quite spread out.[15]

[14] The probability density function of a random variable with normal distribution is $\left[(2\pi)^{0.5}\sigma\right]^{-1} e^{-\frac{1}{2}\left(\frac{\theta-\mu}{\sigma}\right)^2}$, and the constant term is cancelled out for the truncated distribution (32).

[15] We aim to derive a verifiable condition, (33), which leads to the satisfaction of (20)

The next example is the exponential distribution, where $\theta \geq 0$. The probability density function of the truncated exponential distribution in fitting $\theta \in \left[\theta^L, \theta^H\right]$ is

$$f(\theta) = \frac{\lambda e^{-\lambda\theta}}{\int_{\theta^L}^{\theta^H} \lambda e^{-\lambda\theta} d\theta}, \tag{34}$$

where $\lambda > 0$. Using (34), we obtain

$$f(\theta) + \theta f'(\theta) = (1 - \lambda\theta)\, f(\theta).$$

Therefore, if

$$\lambda \leq \frac{1}{\theta^H} \tag{35}$$

holds, then (20) holds for the truncated exponential distribution (34) with the parameter restriction (35). When the annuitants' survival probabilities are distributed according to (34), condition (35) means that the probability density function cannot be very steep.

The last example is the beta distribution, which varies between 0 and 1. To fit $\theta \in \left[\theta^L, \theta^H\right]$, we truncate the beta distribution, with the resulting probability density function as[16]

$$f(\theta) = \frac{\theta^{a-1}(1-\theta)^{b-1}}{\int_{\theta^L}^{\theta^H} \theta^{a-1}(1-\theta)^{b-1} d\theta}, \tag{36}$$

where $a > 0$ and $b > 0$. It can be shown from (36) that

$$f(\theta) + \theta f'(\theta) = \left[a - (b-1)\frac{\theta}{1-\theta}\right] f(\theta). \tag{37}$$

It is easy to see that condition (20) always holds when $0 < b \leq 1$. Since $a - (b-1)\frac{\theta}{1-\theta}$ is decreasing in θ when $b > 1$, its smallest value occurs at $\theta = \theta^H$. Combining the above analysis, we conclude that if

$$b \leq 1 + \frac{a\left(1-\theta^H\right)}{\theta^H}, \tag{38}$$

for all values of θ. It can be seen from the last paragraph of Subsection 4.1 that the problematic case may only happen for the low-risk group when θ is low. For a normal distribution with an inverted-U shape, the probability density function is increasing when θ is relatively low. Thus, condition (20) is likely to still hold even if restriction (33) is not satisfied.

[16] When $\theta \in [0,1]$, the beta distribution is given by $\frac{\theta^{a-1}(1-\theta)^{b-1}}{\int_0^1 \theta^{a-1}(1-\theta)^{b-1} d\theta}$, where $\frac{1}{\int_0^1 \theta^{a-1}(1-\theta)^{b-1} d\theta}$ is known as the beta function.

then (20) holds for all $\theta \in \left[\theta^L, \theta^H\right]$ for the truncated beta distribution (36).

Different parameter values of the beta distribution generate different shapes. One interesting case of the symmetric truncated beta distribution complements the three distributions of survival probability discussed above. When the beta distribution is generated by (36) with the restriction[17]

$$a = b < 1, \tag{39}$$

a U-shaped pattern is observed. It is easy to see from (37) that when the U-shaped $f(\theta)$ is generated by (36) and (39), condition (20) always hold.[18]

In Figure 2, we plot these four distributions that satisfy condition (20).[19] In particular, we present examples for four major cases exhibiting different features.[20] In Panel A (uniform distribution), the probability density function $f(\theta)$ is constant, indicating that annuitants are equally likely to have various levels of survival probability between θ^L and θ^H. In Panel B (normal distribution, with truncation and restricted parameter values), $f(\theta)$ is first increasing and then decreasing, indicating that the annuitants are more likely to be in the middle range of survival probability. In Panel C (exponential distribution, with truncation and parameter restriction), $f(\theta)$ is decreasing, indicating that annuitants are more likely to have low levels of survival probability. In Panel D (beta distribution, with truncation and restricted parameter values), $f(\theta)$ is U-shaped, indicating that the annuitants are more likely to have either very low or very high levels of survival probability.

[Insert Figure 2 here.]

[17] It is well known that the beta distribution is symmetric when $a = b$. Moreover, the symmetric beta distribution generates a U-shaped pattern when $a = b < 1$, a uniform distribution when $a = b = 1$ or a bell-shaped pattern when $a = b > 1$.

[18] While the U-shaped $f(\theta)$ may be less empirically relevant when compared with, for example, the normal distribution with the inverted-U shape, we present this case to shed light on the potentially problematic issue associated with the downward-sloping $f(\theta)$ when θ is low.

[19] In Figure 2, $\left[\theta^L, \theta^H\right] = [0.1, 0.9]$. For the truncated normal distribution, we choose $\mu = 0.5\left(\theta^L + \theta^H\right) = 0.5$ and $\sigma^2 = 0.36$, so as to make the distribution symmetric and satisfy (33). We choose $\lambda = 0.8$ for the truncated exponential distribution. For the truncated beta distribution, we choose $a = 0.5$ and $b = 0.5$ such that $f(\theta)$ is symmetric and U-shaped. The values of $E(\theta)$ of these four distributions are 0.5, 0.5, 0.46 and 0.5, respectively.

[20] We do not present the other major case of increasing probability density function (such as the beta distribution with $a = 5$ and $b = 0.95$), because it is obvious that condition (20) always holds when the probability density function $f(\theta)$ is increasing.

5 Concluding remarks

The equilibrium of an economic model is usually expressed as the fixed point of a function. An example is (9) in this paper, in which the equilibrium price p^* is consistent with the annuitants' choices $\alpha(\theta, p^*)$, which depends on p^* (as well as survival probability θ). While showing the uniqueness of equilibrium is a useful step in many economic studies, it may sometimes be difficult to prove (Abel, 1986, p. 1086; Villeneuve, 2003, p. 534), particularly if the behavioral relationship is complicated.

This paper revisits uniqueness issues of the equilibrium annuity price. Our starting point is the similar equations observed in many papers analyzing insurance products, such as (14) of Abel (1986) about private annuities in the presence of public pension,[21] (8) and (9) of Villeneuve (2003) about the annuity and life insurance markets, (1) of Fang et al. (2008) about the Medigap insurance market, and (32) and (52) of Lau and Zhang (2023) about public annuity plans. We examine whether the equilibrium annuity price is unique or not by focusing on the similar equations in these papers. To illustrate the underlying idea clearly, we choose a simple annuity model with only one source of heterogeneity (survival probability) and do not include other relevant factors.[22]

We obtain two main results, which are summarized in Table 1. Proposition 1 shows that the equilibrium annuity price is unique if the annuitization function $\alpha(\theta, p)$ is multiplicatively separable in survival probability and annuity price, as in (16). This result includes the two well-known special cases (the immediate annuity model with logarithmic utility function and the de-

[21] Equation (14) of Abel (1986) focuses on the difference of annuity revenue and payment, with the equilibrium annuity payout (A^*) defined by

$$\pi(A^*) = \int_{\theta^L}^{\theta^H} M(\theta, A^*)\,\alpha(\theta, A^*)\,f(\theta)d\theta = 0,$$

where $\pi(A)$ is the profit function and $M(\theta, A) = R - \theta A$. This equation appears to be quite different from other equations, but it is easy to show that A^* can be expressed in the same form after rearranging the terms, as follows:

$$R\int_{\theta^L}^{\theta^H} \alpha(\theta, A^*)\,f(\theta)d\theta - A^*\int_{\theta^L}^{\theta^H} \theta\alpha(\theta, A^*)\,f(\theta)d\theta = 0.$$

Note that Abel (1986) specifies the annuity in terms of payout instead of price, and we express (14) of his paper using the notations of this paper. (Abel's (1986) focus on the difference of annuity revenue and payment inspires us to use the profit function (28) to understand the reasons behind the proof of Proposition 2.)

[22] Benartzi et al. (2011) provide an excellent review regarding various factors relevant to annuitization choices.

ferred annuity model in which the annuitization function does not depend on survival probability) in the literature. The main contribution of this paper is the derivation of Proposition 2 that the equilibrium annuity price is unique for all time-separable utility functions, provided that a sufficient condition (20) is satisfied. Interestingly, this sufficient condition depends only on the distribution of survival probabilities. We further show that this condition is not restrictive and it is satisfied by many distributions, including the commonly-used uniform distribution and normal distribution (with appropriate truncation to fit the survival probabilities within the bounded interval $\left[\theta^L, \theta^H\right]$ and with parameter restriction).

[Insert Table 1 here.]

The contributions of Proposition 2 can be viewed from two angles: the results and the method. We prove uniqueness of equilibrium in a simple two-period model with non-exclusive private annuities and asymmetric information about (continuous) survival probability. Strictly speaking, the *results* of Proposition 2 are applicable in this environment only. However, we believe that the underlying idea of the proof, especially that related to the direct and indirect effects of a change in annuity price on the budget balance of annuity providers through the responses of high-risk and low-risk annuitants, and that based on the link between $\frac{\partial\alpha(\theta,p)}{\partial p}$ and $\frac{\partial\alpha(\theta,p)}{\partial\theta}$, are also useful in other settings involving annuity and insurance products. As an example, the *method* used in the proof of Proposition 2 has already been successfully applied to show that the equilibrium price is unique for the immediate annuity market in an economy with risks in both longevity and income (Brugiavini, 1993, Section 4), provided that a sufficient condition about the annuitants' survival probability density function conditional on income is satisfied. Applications to other models will be left to future studies.

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Figure 1: The $J(p)$ function and its fixed point

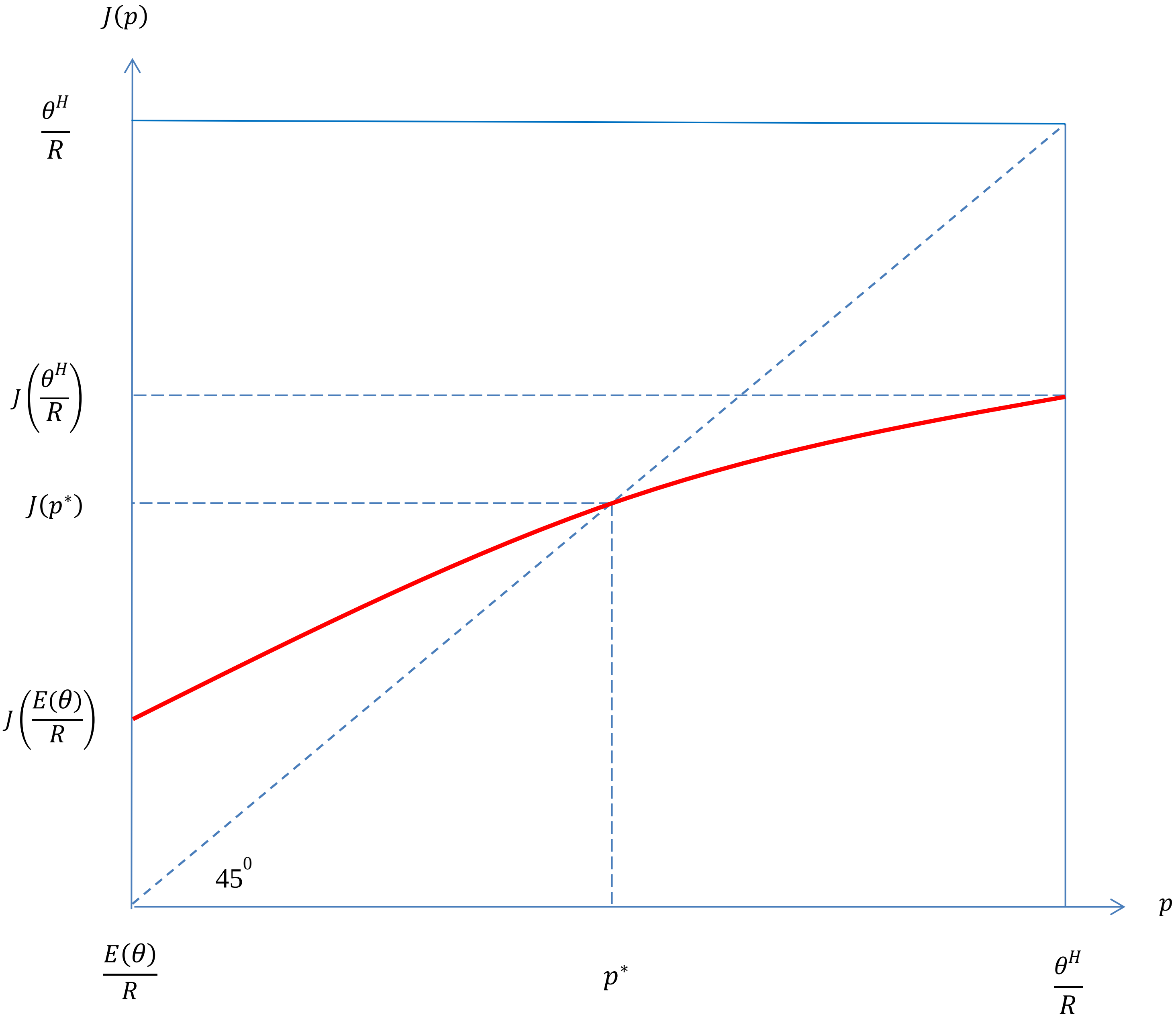
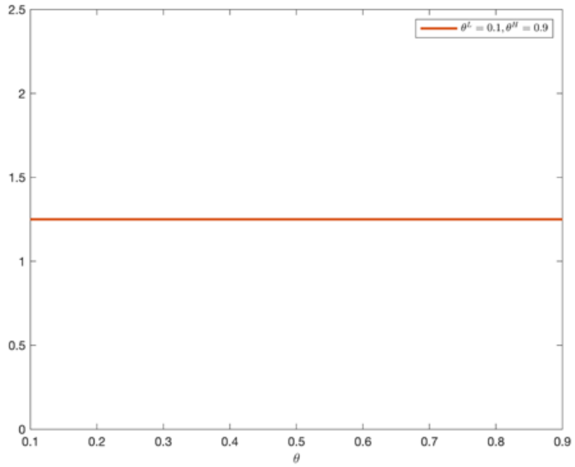
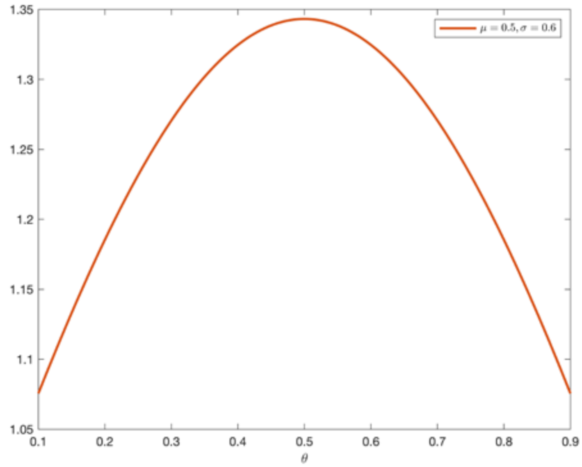


Figure 2: Four probability density functions satisfying Proposition 2

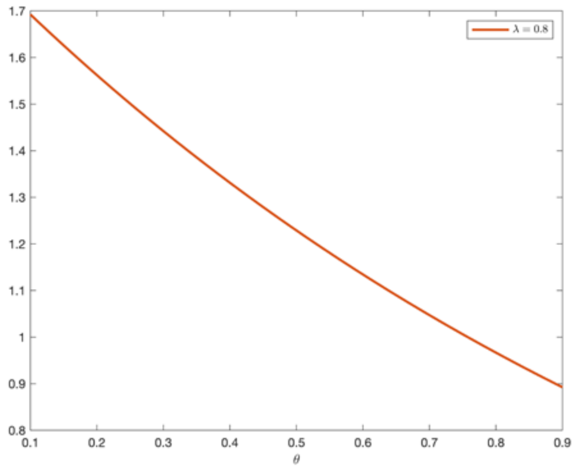
Panel A: Uniform distribution



Panel B: Truncated normal distribution



Panel C: Truncated exponential distribution



Panel D: Truncated beta distribution

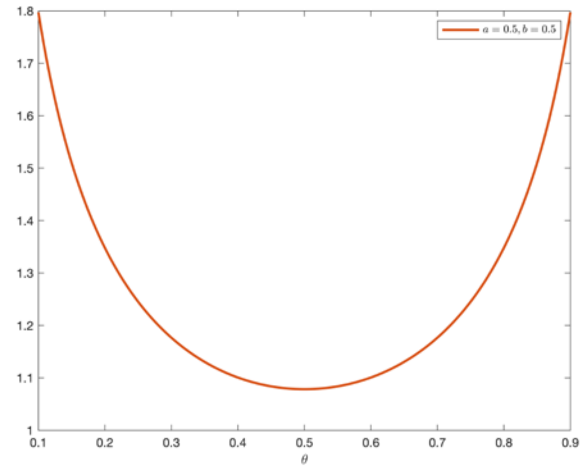


Table 1: Multiplicatively separable and more general annuitization functions

	$\alpha(\theta, p)$	$J(p)$	Uniqueness of p^*
Immediate annuity market when annuitants have logarithmic utility functions (Abel, 1986; Brown, 2003)	$=\alpha_1(\theta)\alpha_2(p)$	$=\dfrac{\int_{\theta^L}^{\theta^H} \theta\alpha_1(\theta) f(\theta) d\theta}{R\int_{\theta^L}^{\theta^H} \alpha_1(\theta) f(\theta) d\theta}$	Always
Deferred annuity market (Brugiavini, 1993)	$=\alpha_2(p)$	$=\dfrac{E(\theta)}{R}$	Always
Immediate annuity market when annuitants have general time-separable utility functions	Depends on θ and p, but is not multiplicatively separable	$=\dfrac{\int_{\theta^L}^{\theta^H} \theta\alpha(\theta, p) f(\theta) d\theta}{R\int_{\theta^L}^{\theta^H} \alpha(\theta, p) f(\theta) d\theta}$	Sufficient condition: (20)